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$$RP(t) = RP_0 + \sum_{k=1} RP_k t^k$$

$$tA(t)P(t) = t[A_0P_0 + t(A_1P_0 + A_0P_1) + \dots]$$

$$= t\sum C_k t^k = \sum_{k=1} C_{k-1} t^k, \quad C_k = \sum_{j=0}^{k} A_{k-j}P_j$$

$$\therefore \lambda P(t) + tP'(t) = RP(t) + tA(t)P(t)$$

$$\Downarrow$$

$$(\lambda I - R)P_0 + \sum_{k=1}[\lambda P_k + kP_k - RP_k - C_{k-1}]t^k = 0$$

this implies that λ is an eigenvalue of R, P_0 is the corresponding ~~eig~~ eigenvector, and

$$[(\lambda + k)I - R]P_k = C_{k-1}$$

$$P_k = [(\lambda+k)I - R]^{-1} C_{k-1}$$ since $\lambda + k$ is not an eigenvalue.

$$\because kP_k = -(\lambda I - R)P_k + C_{k-1}$$

$$\therefore |k||P_k| \le |\lambda I - R||P_k| + |C_{k-1}|$$

choose k so large that, $\frac{k}{2} \geq |\lambda I - R|$

$$\therefore |k||P_k| \le 2|C_{k-1}|$$

since $A(t)$ is a convergent series, let $0 < |t| < r < \rho$,

$$|A_j| r^j < M.$$

$$|k||P_k| \le 2\sum_{j=0}^{k-1} |A_{k-1-j}||P_j| \le 2r^{1-k}\sum_{j=0}^{k-1}|P_j|r^j$$

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let $d_k = \|P_k\|$, and

$k d_k = 2M r^{1-k} \sum_{j=0}^{k-1} d_j r^j$, $\Rightarrow \|P_k\| \leq d_k$

thus, $(k+1) d_{k+1} = 2M r^{-k} \sum_{j=0}^{k} d_j r^j$

$k d_k = 2M r^{1-k} \sum_{j=0}^{k-1} d_j r^j \quad / \, r$

$$(k+1) d_{k+1} = r^{-1} k d_k + 2M r^{-k} d_k r^k$$
$$= (2M + r^{-1} k) d_k$$

$$\left| \frac{d_{k+1}}{d_k} \frac{t^{k+1}}{t^k} \right| = \left| \frac{2M}{k+1} + \frac{k}{k+1} \frac{1}{r} \right| |t| \xrightarrow[k \to \infty]{} \frac{|t|}{r} < 1$$

$\therefore$ $P(t)$ converges absolutely. $\square$

from the previous theorem, let λ be $Re(\lambda) = \max[Re(\lambda_j)]$

there is a solution $X(t) = |t-\tau|^{\lambda} P(t)$.

theorem:

$R \in M_n(\mathbb{C})$, has n distinct eigenvalues, $\lambda_1 \cdots \lambda_n$, no two differ by an integer, then the basis

$$X = (x_1 \cdots x_n), \quad x_i = |t-\tau|^{\lambda_i} P_i(t)$$

is the solution.

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proof:

By the previous theorem, if $P_i(\tau)$ is an eigenvector of eigenvalue λ_i, the solution exists.

$$X(t) = P(\tau)\,|t-\tau|^{S}$$

$$P = (P_1 \ \cdots \ P_n) \qquad S = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

since $\det(P(\tau)) \neq 0$, $X(\tau)$ is invertible, so it is a basis. $\square$

Example:

$$R = \frac{1}{6}\begin{pmatrix} 5 & -6 \\ 4 & -6 \end{pmatrix} \qquad A = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\lambda - \frac{5}{6}\right)(\lambda + 1) + \frac{24}{36} = 0$$

$$\lambda^2 + \frac{1}{6}\lambda - \frac{1}{6} = 0 \qquad \therefore \lambda_1 = -\frac{1}{2}$$

$$\left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{3}\right) = 0, \qquad \lambda_2 = \frac{1}{3}$$

$$\lambda_1 : \frac{1}{6}(5a - 6b) = -\frac{1}{2}a \Rightarrow a = \frac{3}{4}b$$

$$\frac{1}{6}(4a - 6b) = -\frac{1}{2}b \Rightarrow a = \frac{3}{4}b \qquad \therefore P_0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\lambda_2 : \frac{1}{6}(5a - 6b) = \frac{1}{3}a \Rightarrow a = 2b$$

$$\frac{1}{6}(4a - 6b) = \frac{1}{3}b \Rightarrow a = 2b \qquad \therefore Q_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore U(t) = |t|^{-\frac{1}{2}} P(t) \qquad P(t) = P_0 + \sum_{k=1} P_k t^k$$

$$V(t) = |t|^{1/3} Q(t) \qquad Q(t) = Q_0 + \sum_{k=1} Q_k t^k$$

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$[(\lambda+k)I - R]P_k = C_{k-1}$

$k=1$, $[(\lambda+1)I - R]P_1 = A_0 P_0 = 3P_0$

note that $[(\lambda+k)I - R]P_0 = (\lambda+k)P_0 - \lambda P_0 = kP_0$

$\therefore \frac{P_0}{k} = [(\lambda+k)I - R]^{-1} P_0$

$\therefore P_1 = [(\lambda+1)I - R]^{-1} 3P_0 = 3P_0$

$P_2 = [(\lambda+2)I - R]^{-1}(A_0 P_1 + A_1 P_0) = [(\lambda+2)I - R]^{-1} 3P_1$

$= \frac{9}{2} P_0$

$P_3 = \frac{27}{2\cdot 3} P_0$

$\vdots$

$P_k = \frac{3^k}{k!} P_0$, similarly $Q_k = \frac{3^k}{k!} Q_0$

$\therefore P(t) = e^{3t} P_0 \quad Q(t) = e^{3t} Q_0$

$\therefore X(t) = (u(t) \ v(t)) = e^{3t}\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} |t|^{-\frac{1}{2}} & 0 \\ 0 & |t|^{\frac{1}{3}} \end{pmatrix}$

what if $\lambda_2 - \lambda_1$ is an integer?

Examples:

① $tX' = RX$, $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda(\lambda-1) = 0$, $\lambda_1 = 0$, $\lambda_2 = 1$

by inspection, $X(t) = (u(t) \ v(t)) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$

② $tX' = [R + A_0 t]X$, $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\lambda_1 = 1$, $\lambda_2 = 0$

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for λ_2, eigenvector $P_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\because \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

however, $[(\lambda_2+1)I - R]P_1 = A_0 P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \text{no solution,}$$

by inspection, $X(t) = \begin{pmatrix} t & t\log|t| \\ 0 & 1 \end{pmatrix}$, $X' = \begin{pmatrix} 1 & \log|t|+1 \\ 0 & 0 \end{pmatrix}$

Cayley-Hamilton theorem (another prove)

$$p_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$$
$$= (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_k)^{m_k}$$

from primary decomposition theorem,

$\forall \xi = \xi_1 + \dots + \xi_k$, where

$(A - \lambda_i I_n)^{m_i}\xi_i = 0$, $i = 1, \dots, k$

let $p_A(A) = (A-\lambda_1 I_n)^{m_1} \cdots (A-\lambda_k I_n)^{m_k}$, all factors commute

$\therefore p_A(A)\xi = (A-\lambda_1 I_n)^{m_1} \cdots (A-\lambda_k I_n)^{m_k}(\xi_1 + \dots + \xi_k)$

$\equiv 0$

$\therefore p_A(A) = 0 = A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0$

$UA - BU = C$

$UA - BU$ is a linear mapping.
if A and B have no common eigenvalue, the only solution of $UA - BU = 0$ is $U \equiv 0$, thus $UA - BU = C$ has a unique solution $\forall C$.

proof:

note that if $UA = BU$, $UA^2 = BUA = B^2U$,

$\therefore UA^k = B^kU$

let $p_B(A) = (A - \mu_1 I_n)^{m_1} \cdots (A - \mu_p I_n)^{m_p}$

since μ_i is not eigenvalue of A, all factors are invertible.

$$\begin{aligned} U p_B(A) &= U(A - \mu_1 I_n)^{m_1} \cdots (A - \mu_p I_n)^{m_p} \\ &= U(A^n + b_{n-1}A^{n-1} + \cdots + b_1 A + b_0) \\ &= (B^n + b_{n-1}B^{n-1} + \cdots + b_1 B + b_0)U \\ &= p_B(B)U \equiv 0 \end{aligned}$$

$\therefore U \equiv 0$, since $p_B(A)^{-1}$ exists.

theorem:

R with no eigenvalues difference by any integers, $X(t) = P(t)|t - \tau|^R$ is a solution to

$$X' = \left[\frac{R}{t-\tau} + A(t)\right]X$$

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$$\boxed{P(t)=\sum P_k (t-\tau)^k, \quad |t-\tau|<\rho, \quad P_0 = I_n}$$

suppose $X(t)=P(t)|t|^R$ is a solution basis.

$$t X' = [R + tA(t)]X$$

$$t[P'(t)|t|^R + P(t) R |t|^R / t] = R P(t)|t|^R + t A(t) P(t)|t|^R$$

$|t|^R$ is invertible for $|t|>0$,

$$\therefore \quad tP'(t) + P(t)R = RP(t) + tA(t)P(t)$$

[note $\because |t|^R = I + R\log|t| + \frac{R^2}{2!}(\log|t|)^2 + \dots$,

$$\therefore \frac{d}{dt}|t|^R = \frac{R}{t}|t|^R \quad]$$

similar to the proof of previous theorem,

$$tP'(t) = t\sum kP_k t^{k-1} = \sum_{k=1} kP_k t^k$$

$$P(t)R = P_0 R + \sum_{k=1} P_k R t^k, \quad RP(t) = RP_0 + \sum_{k=1} RP_k t^k$$

$$tA(t)P(t) = t\sum C_k t^k = \sum_{k=1} C_{k-1} t^k, \quad C_k = \sum_{j=0} A_{k-j} P_j$$

hence, $P_0 R = R P_0 \Rightarrow P_0 = I_n$

$$P_k[kI_n + R] - RP_k = C_{k-1}$$

let μ be an eigenvalue of $kI_n + R$,

$$\det[\mu I_n - kI_n - R] = \det[(\mu - k)I_n - R] \Rightarrow \underbrace{\mu - k = \lambda}$$

eigenvalue of R

since no two eigenvalues differ by an integer, no common eigenvalues between $kI_n + R$ and R

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$\therefore \exists P_k$ such that $P_k[kI_n + R] - RP_k = C_{k-1}$

suppose $P(t)$ is convergent on $|t| < \rho$, continuity of $P(t)$ and $P_0 = I_n$ imply that $P(t)$ is invertible near $t=0$, $|t|^R$ is also invertible near $t=0$, thus X is invertible.

$\therefore X(t)$ is a basis for $0 < |t| < \rho$

proof of convergence of $P(t)$ is the same as before

$\square$

Example:

$$X' = \left[\frac{R}{t} + A(t)\right]X, \quad R = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}, \quad A(t) = tI_2$$

$$(\lambda - 1)(\lambda + 3) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

$\therefore \lambda = -1$

$$\therefore \begin{matrix} a + 2b = -a \\ -2a - 3b = -b \end{matrix} \Rightarrow a = -b, \quad Q_1 = \begin{pmatrix} a \\ -a \end{pmatrix}$$

for ~~the~~ a second eigenvector,

$$(\lambda I_2 - R)^2 Q_2 = 0$$

$$\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}^2 Q_2 = 0 \qquad \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore 0 Q_2 = 0$$

we can choose any vector independent from Q_1 to be second eigenvector, let $Q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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Jordan form $= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ $\because$ only 1 linearly independent vector Q_1.

$QJQ^{-1} = \begin{pmatrix} a & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & a \end{pmatrix}\frac{1}{a}$

$= \begin{pmatrix} a & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} -1+a & a \\ -a & -a \end{pmatrix}\frac{1}{a}$

$= \begin{pmatrix} a^2-a & a^2 \\ -a^2+a & -a^2-a \end{pmatrix}\cdot\frac{1}{a} = \begin{pmatrix} a-1 & a \\ 1-a & -1-a \end{pmatrix}$, $\therefore a=2$

$= R$

$\therefore Q_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

from the previous theorem, $X(t) = P(t)|t|^R$, $P_0 = I_2$,

it is tedious to compute P_k, so instead let

$X(t) = U(t)e^{t^2/2}$

$X' = U'e^{t^2/2} + tUe^{t^2/2} = U'e^{t^2/2} + tX$

$\therefore U' = \frac{R}{t}U$

$\therefore U = |t|^R = e^{R\log|t|}$

$\therefore X(t) = e^{t^2/2}|t|^R$, $P(t) = e^{t^2/2}I_2$

since $|t|^R = Q|t|^J Q^{-1}$, $|t|^J = I_2 + J\log|t| + \frac{J^2}{2!}(\log|t|)^2 + \cdots$

$= I_2 + \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\log|t| + \frac{1}{2!}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}(\log|t|)^2 + \cdots$

change it to basis $Y(t)$

$Y(t) = X(t)Q$

$= e^{t^2/2}Q|t|^J$

$= |t|^{-1}e^{t^2/2}\begin{pmatrix} 2 & 2\log|t| \\ -2 & 1-2\log|t| \end{pmatrix}$

$|t|^J = |t|^{-1}\begin{pmatrix} 1 & \log|t| \\ 0 & 1 \end{pmatrix}$

Singular points of the first kind: general case

when two eigenvalues differ by an integer, they can be switched to the same by change of variables.

$$\forall R,\quad R = Q J Q^{-1},$$

$$J = Q^{-1} R Q$$

let's write J as $\mathrm{diag}(R_1, R_2)$,

$$R_1 = \begin{pmatrix} \lambda_1 & * & * & * \\ & \ddots & * & * \\ & & \ddots & * \\ 0 & & & \lambda_1 \end{pmatrix} \qquad R_1 \in M_{n_1}(\mathbb{C})$$

let $X = QY$

$$X' = \left[\frac{R}{t} + A(t)\right] X, \quad \text{assuming } \tau = 0$$

$$QY' = \left[\frac{QJQ^{-1}}{t} + A(t)\right] QY$$

$$Y' = \left(\frac{J}{t} + B(t)\right) Y, \quad B(t) = Q^{-1} A Q$$

note that when $n = 1$, if $y = tz$

$$z + tz' = \left(\frac{j}{t} + b(t)\right) tz$$

$$z' = \left(\frac{j-1}{t} + b(t)\right) z$$

j is transformed to $j-1$!

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thus, in general case, set

$$Y = UZ, \quad U = \mathrm{diag}(t I_{m_1}, I_{n-m_1})$$
$$U^{-1} = \mathrm{diag}(t^{-1} I_{m_1}, I_{n-m_1})$$

$$\therefore Z' = U^{-1}\left[\frac{JU}{t} - U' + B(t)U\right]Z$$

for the first m_1 and m_1 row columns,

$$U^{-1}\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_1 \end{pmatrix} U = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_1 \end{pmatrix}$$

$$\therefore U^{-1} J U = J = \mathrm{diag}(R_1, R_2)$$

$$U^{-1}U' = \mathrm{diag}(t^{-1} I_{m_1}, I_{n-m_1}) \cdot \mathrm{diag}(I_{m_1}, 0)$$
$$= \mathrm{diag}(t^{-1} I_{m_1}, 0)$$

$$B(t) = \sum B_k t^k, \quad B_k = \begin{array}{c} \begin{pmatrix} (B_k)_{11} & (B_k)_{12} \\ (B_k)_{21} & (B_k)_{22} \end{pmatrix} \begin{array}{l} \}m_1 \\ \}n-m_1 \end{array} \\ \underbrace{\quad\quad}_{m_1} \quad \underbrace{\quad\quad}_{n-m_1} \end{array}$$

$$U^{-1} B_k U = \mathrm{diag}(t^{-1} I_{m_1}, I_{n-m_1}) \begin{pmatrix} (B_k)_{11} & (B_k)_{12} \\ (B_k)_{21} & (B_k)_{22} \end{pmatrix} \mathrm{diag}(t I_{m_1}, I_{n-m_1})$$

$$= \mathrm{diag}(t^{-1} I_{m_1}, I_{n-m_1}) \begin{pmatrix} t(B_k)_{11} & (B_k)_{12} \\ t(B_k)_{21} & (B_k)_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (B_k)_{11} & t^{-1}(B_k)_{21} \\ t(B_k)_{21} & (B_k)_{22} \end{pmatrix}$$

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$\therefore C(t) = U^{-1} B U$

$= t^{-1} C_{-1} + \sum C_k t^k$

$C_{-1} = \begin{pmatrix} 0 & (B_0)_{12} \\ 0 & 0 \end{pmatrix} \quad C_0 = \begin{pmatrix} (B_0)_{11} & (B_1)_{12} \\ 0 & (B_0)_{22} \end{pmatrix}$

$C_k = \begin{pmatrix} (B_k)_{11} & (B_{k+1})_{12} \\ (B_{k-1})_{21} & (B_k)_{22} \end{pmatrix}$

$\therefore z' = [K t^{-1} + C(t)] z$

$K = \begin{pmatrix} R_1 - I_{m_1} & (B_0)_{12} \\ 0 & R_2 \end{pmatrix}$

the eigenvalues $\lambda_1, \cdots, \lambda_k$ have been shifted to $\lambda_1 - 1, \cdots, \lambda_k$ by $X = QY = QUz$

thus, by finite change of variables,

$X = Q_1 U_1 Q_2 U_2 \cdots Q_s U_s W$

{ e.g. $J = Q_2^{-1} K Q_2$

$W' = [S t^{-1} + D(t)] W$

$W(t) = V(t)|t|^S$, S has no two eigenvalues difference by an integer.

$\therefore X(t) = P(t)|t|^S = Q_1 U_1 \cdots Q_s U_s V(t) |t|^S$

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Example:

$X' = \left(\frac{R}{t} + A\right)X$, $R = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$, $A = \begin{pmatrix} 3 & -1 \\ 0 & 3 \end{pmatrix}$

$(\lambda+1)(\lambda+2) = 0$, $\therefore \lambda_1 = -1$, $\lambda_2 = -2$

$\begin{matrix} -a+b = -a \\ -2b = -b \end{matrix} \Rightarrow Q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} -a+b = -2a \\ -2b = -2b \end{matrix} \Rightarrow Q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Q^{-1} = Q$

$J = Q^{-1}RQ = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

let $X = QY$,

$Y' = \left(\frac{J}{t^{-1}} + B\right)Y$, $B = Q^{-1}AQ = Q^{-1}\begin{pmatrix} 3 & 4 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

let $Y = VZ = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} Z$

$Z' = \left(\frac{K}{t} + C\right)Z$, $\because \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & t^{-1} \\ 0 & 3 \end{pmatrix}$

$\therefore Z(t) = V(t)\,t^{K}$, $t > 0$ $\quad \therefore C = 3I_2$, $K = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$

although coeff. of $V(t)$ can be computed by substitution, note that

let $Z(t) = e^{3t}W(t)$

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$$z' = 3z + e^{3t}w' \Rightarrow w' = Kt^{-1}w$$

$$\therefore \ w = t^{K}$$

$$\therefore \ z = e^{3t}t^{K}$$

$$X(t) = QUz = e^{3t}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}t^{K}$$

$$= e^{3t}\begin{pmatrix} t & 1 \\ 0 & -1 \end{pmatrix}t^{K}$$

$$t^{K} = I_2 + K\log t + \frac{K^2}{2!}\log t + \dots \qquad \because K^2 = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \log t \\ 0 & 1 \end{pmatrix}t^{-2} \qquad = \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix} \to \text{induction.}$$

$$\therefore \ X(t) = e^{3t}\begin{pmatrix} t & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & \log t \\ 0 & 1 \end{pmatrix}t^{-2} = t^{-2}e^{3t}\begin{pmatrix} t & 1 + t\log t \\ 0 & -1 \end{pmatrix}$$

second answer:

Singular points of the second kind

$$(t-\tau)^{r+1}X' = A(t)X, \quad r>0$$

Consider the simplest example where $A(t)=A$, a constant.

$$\therefore (t-\tau)^{r+1}X' = AX$$

$$X(t) = e^{-\frac{A}{r}(t-\tau)^{-r}}, \quad \because X' = \frac{d}{dt}\left(1-\frac{A}{r}(t-\tau)^{-r}+\frac{A^2}{2r^2}(t-\tau)^{-2r}+\dots\right)$$

$$= A(t-\tau)^{-r-1} - \frac{A^2}{r}(t-\tau)^{-2r-1}+\dots$$

$$= A(t-\tau)^{-r-1}e^{-\frac{A}{r}(t-\tau)^{-r}}$$

solution basis

Example:

$$t^2X' = \left[\begin{pmatrix}0&0\\0&1\end{pmatrix}+\begin{pmatrix}0&1\\-1&-2\end{pmatrix}t\right]X, \quad X=\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

$$t^2x_1' = tx_2 \qquad t^2x_2' = x_2-(x_1+2x_2)t$$

let $x_1 = \sum x_{1k}t^k$, $x_2 = \sum x_{2k}t^k$

thus, $t^2\sum kx_{1k}t^{k-1} = t\sum x_{2k}t^k$

$$\therefore \sum_{k=1} x_{2k-1}t^k = \sum_{k=1} kx_{1k}t^{k+1} = \sum_{k=2}(k-1)x_{1k-1}t^k$$

$$\therefore x_{20}=0$$

another equation, $\sum_{k=2}(k-1)x_{2k-1}t^k = \sum x_{2k}t^k - \sum x_{1k}t^{k+1} - 2\sum x_{2k}t^{k+1}$

$$= x_{21}t - x_{10}t$$

$$+\sum_{k=2}(x_{2k}-x_{1k-1}-2x_{2k-1})t^k$$

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$$\therefore \quad k x_{1k} = x_{2k}$$

$$(k-1) x_{2k-1} = x_{2k} - \frac{x_{2k-1}}{k-1} - 2 x_{2k-1}$$

$$x_{10} = x_{21}$$

$$x_{20} = 0$$

$$\to x_{2k} = \left[k-1 + \frac{1}{k-1} + 2\right] x_{2k-1}$$

$$= \left(\frac{k-1^2+1}{k-1}\right) x_{2k-1} = \frac{k^2}{k-1} x_{2k-1}$$

if setting $x_{10} = x_{21} = 1$,

$$x_{22} = 4 = 2 \cdot 2!$$

$$x_{23} = \frac{3 \cdot 3}{2} \cdot 2 \cdot 2! = 3 \cdot 3!$$

$$\Rightarrow x_{2k} = k \cdot k!$$

$$x_{1k} = k!$$

$$\therefore x_1 = \sum (k)! \, t^k$$

$$x_2 = \sum k (k)! \, t^k$$

diverging series !!
(for $t \neq 0$)

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Single equations with singular points

Equations of order n

$$c_n(t)x^{(n)} + \cdots + c_1(t)x' + c_0(t)x = 0$$

of special case,

$$(t-\tau)^n x^{(n)} + (t-\tau)^{n-1} a_{n-1}(t) x^{(n-1)} + \cdots + (t-\tau) a_1(t) x^{(1)} + a_0(t)x = 0$$

if $a_0, \dots, a_{n-1}$ are analytic at τ, and at least one of them is not zero, then τ is regular singular point.

by change of variables,

$$\begin{aligned} y_0 &= x \\ y_1 &= y_0' = x' \\ &\vdots \\ y_{n-1} &= y_{n-2}' = x^{(n-1)} \end{aligned}$$

it becomes

$$(t-\tau)^n Y' = A(t)Y$$

$$A(t) = \begin{pmatrix} 0 & (t-\tau)^n & 0 & & 0 \\ 0 & 0 & (t-\tau)^n & & 0 \\ & & \ddots & 0 & (t-\tau)^n \\ 0 & & & & \\ -a_0(t) & \cdots & & & -a_{n-1}(t)(t-\tau)^{n-1} \end{pmatrix}$$

if all $a_j(t) = (t-\tau)^{n-1-j} b_j(t)$, where $b_j(t)$ are analytic at τ, then it becomes $(t-\tau)Y' = \tilde{A}(t)Y$, thus τ is a singular point of the first kind.

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let $z_j(t) = (t-\tau)^{j-1} x^{(j-1)}(t)$, $j = 1, \dots, n$

$(t-\tau) z_j'(t) = (j-1)(t-\tau)^{j-1} x^{(j-1)}(t) + z_{j+1}(t)$

$= (j-1) z_j(t) + z_{j+1}(t)$

$\therefore (t-\tau) z_n' = (n-1) z_n(t) + (t-\tau)^n x^{(n)}(t)$

$= -a_0(t) z_1(t) - a_1(t) z_2(t) - \dots + [(n-1) - a_{n-1}(t)] z_n(t)$

hence,

$$\boxed{\begin{gathered}(t-\tau) z' = B(t) z \\ B(t) = \begin{pmatrix} 0 & 1 & & & & 0 \\ & 1 & 1 & & & \\ 0 & & 2 & 1 & & \\ & & & \ddots & n-2 & 1 \\ -a_0(t) & -a_1(t) & \dots & & -a_{n-2}(t) & (n-1) - a_{n-1}(t) \end{pmatrix} \\ z(t) = \begin{pmatrix} x(t) \\ (t-\tau) x'(t) \\ \vdots \\ (t-\tau)^{n-1} x^{(n-1)} t \end{pmatrix}\end{gathered}}$$

write $A(t) = \begin{pmatrix} 0 & 0 & & & 0 \\ 0 & 0 & 0 & & \\ & & 0 & 0 & \ddots \\ & 0 & & & \\ -(a_0(t) - a_0(\tau)) & & \dots & & -(a_{n-1}(t) - a_{n-1}(\tau)) \end{pmatrix}$

$R(\tau) = \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 1 & 1 & & \\ & & 2 & 1 & \\ & 0 & & \ddots & \\ -a_0(\tau) & -a_1(\tau) & & \dots & (n-1) - a_{n-1}(\tau) \end{pmatrix}$

then, $\dfrac{A(t)}{t-\tau}$ is analytic at τ because $a_i(t)$ is analytic, assuming that τ is a regular singular point.

$$\therefore \quad z' = \left[\frac{R}{t-\tau} + A(t)\right] z, \quad \text{where } A(t) = \frac{A(t)}{t-\tau}$$

$$p_R(\lambda) = \det(\lambda I_n - R)$$

$$= \det\begin{pmatrix} \lambda & -1 & & & \\ & \lambda-1 & -1 & & \\ & & \lambda-2 & -1 & \\ & & & \ddots & -1 \\ a_0(\tau) & \cdots & \cdots & \cdots & \lambda-(n-1)+a_{n-1}(\tau) \end{pmatrix}$$

expansion on first column

$$= \lambda \det(\lambda I_{n-1} - R') + a_0(\tau)$$

$\because$ if n is odd, $(-1)^{2m}$; $\begin{pmatrix} + \\ + \end{pmatrix}$

if n is even, $(-1)^{2m+1} \cdot (-1)$

$$= \lambda(\lambda-1)\det(\lambda I_{n-2} - R'') + a_0(\tau) + a_1(\tau)\lambda$$

$$\vdots$$

$$= \lambda(\lambda-1)\cdots(\lambda-n+1) + \lambda(\lambda-1)\cdots(\lambda-n+2)a_{n-1}(\tau)$$
$$+ \cdots + \lambda a_1(\tau) + a_0(\tau)$$

if $n=1$, note that

$$(t-\tau)x'(t) = [-a_0(\tau) - (a_0(t) - a_0(\tau))]x$$

$$x'(t) = \left[\frac{-a_0(\tau)}{t-\tau} - \frac{a_0(t)-a_0(\tau)}{t-\tau}\right]x$$

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$$\therefore R = -a_0(\tau)$$

$$\therefore P_R(\lambda) = \lambda + a_0(\tau)$$

now, suppose α is the eigenvector with eigenvalue λ.

$$R\alpha = \lambda\alpha$$

$$\therefore \alpha_2 = \lambda\alpha_1, \quad \alpha_2 + \alpha_3 = \lambda\alpha_2, \quad 2\alpha_3 + \alpha_4 = \lambda\alpha_3,$$

$$\cdots \quad \underbrace{-a_0(\tau)\alpha_1 - a_1(\tau)\alpha_2 - \cdots + [(n-1) - a_{n-1}(\tau)]\alpha_n = \lambda\alpha_n}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$\alpha_j = \lambda(\lambda-1)\cdots(\lambda-j+2)\alpha_1, \quad j = 2, \cdots, n$$

and the last equation implies $P_R(\lambda)\alpha_1 \equiv 0$, $\therefore \alpha_1$ is arbitrary

since $\mathcal{E}(R,\lambda)$ consists of a single vector, i.e.,

$$\alpha = \begin{pmatrix} 1 \\ \lambda \\ \lambda(\lambda-1) \\ \vdots \\ \lambda(\lambda-1)\cdots(\lambda-n+2) \end{pmatrix}$$

$$\therefore \dim(\mathcal{E}(R,\lambda)) = 1$$

as the equation becomes the form of $(t-\tau)z' = \left[\frac{R}{t-\tau} + A(t)\right]z$,

theorems of first kind of singular points apply.

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① if λ is a root of $p_R(\lambda)$, and

$$Re(\lambda) = \max(Re(\lambda_j)), \quad j = 1, \cdots, k.$$

$$x(t) = |t-\tau|^{\lambda} p(t), \quad p(t) = \sum p_k (t-\tau)^k, \quad 0 < |t-\tau| < \rho$$

$$p_0 \neq 0$$

② if $\lambda_1, \cdots, \lambda_n$ do not differ by an integer,

then

$$x_j(t) = |t-\tau|^{\lambda_j} p_j(t), \quad 0 < |t-\tau| < \rho, \quad p_j(\tau) \neq 0$$

③ if $\lambda_1, \cdots, \lambda_k$ do not differ by an integer, with multiplicities $m_1, \cdots, m_k$, then for each root λ, with multiplicity m, the solutions are

$$y_1 = |t-\tau|^{\lambda} p_1(t)$$

(see 11/3/2019 note)

$$y_2 = |t-\tau|^{\lambda} [p_1(t) \log|t-\tau| + p_2(t)]$$

$$\vdots$$

$$y_m = |t-\tau|^{\lambda} \left[\frac{p_1(t)}{(m-1)!} [\log|t-\tau|]^{m-1} + \cdots + p_m(t)\right]$$

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The Euler equation

$$Lx = t^n x^{(n)} + a_{n-1} t^{n-1} x^{(n-1)} + \dots + a_1 t x' + a_0 x = 0$$

$a_0, \dots, a_{n-1}$ are constants, $\in \mathbb{C}$.

by transforming it to $z' = \left[\frac{R}{t-\tau} + A(t)\right] z$ form,

$$z' = \frac{R}{t} z, \qquad R = \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 1 & 1 & & \\ & 0 & 2 & 1 & \\ 0 & & \ddots & & \\ & & & n-2 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & (n-1)-a_{n-1} \end{pmatrix}$$

$$\therefore z = |t|^R$$

alternatively,

let's apply the operator L to t^λ by noting that

$$Dt^\lambda = \frac{d}{dt} t^\lambda = \lambda t^{\lambda-1}$$

$$\therefore t D t^\lambda = \lambda t^\lambda, \quad t^2 D^2 t^\lambda = \lambda(\lambda-1) t^\lambda, \quad t^i D^i t^\lambda = \lambda \cdots (\lambda-i+1) t^\lambda$$

$$L(t^\lambda) = [\lambda(\lambda-1)\dots(\lambda-n+1) + a_{n-1} \cdot \lambda(\lambda-1)\dots(\lambda-n+2) + \dots + a_1 \lambda + a_0] t^\lambda$$

$$\therefore \boxed{L(t^\lambda) = p_R(\lambda) t^\lambda}$$

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$$\frac{\partial^l}{\partial\lambda^l} L(t^\lambda) = L\left(\frac{\partial^l}{\partial\lambda^l} t^\lambda\right) = L\left(\frac{\partial^l}{\partial\lambda^l} e^{\lambda\log t}\right) = L\left(t^\lambda(\log t)^l\right)$$

$$= \frac{\partial^l}{\partial\lambda^l}\left[P_R(\lambda)t^\lambda\right]$$

$$= P_R^{(l)}(\lambda)t^\lambda + l P_R^{(l-1)}(\lambda)t^\lambda\log t + \cdots$$

$$+ l P_R'(\lambda)t^\lambda(\log t)^{l-1} + P_R(\lambda)t^\lambda(\log t)^l$$

if the multiplicity of λ_1 is m_1, thus

$$P_R(\lambda_1) = P_R'(\lambda_1) = \cdots = P_R^{(m_1-1)}(\lambda_1) = 0, \quad P_R^{(m_1)}(\lambda_1) \neq 0$$

$$\therefore L\left(t^\lambda(\log t)^l\right) = \left[P_R^{(l)}(\lambda) + P_R^{(l-1)}(\lambda)\log t + \cdots + P_R(\lambda)(\log t)^l\right]t^\lambda$$

implies that if $\lambda = \lambda_1$, $l \leq m_1 - 1$,

$L(t^{\lambda_1}(\log t)^l) = 0$, $t^{\lambda_1}(\log t)^l$ is a solution.

$$\boxed{\text{solution basis of Euler equation: } t^{\lambda_1},\ t^{\lambda_1}\log t,\ \ldots,\ t^{\lambda_1}(\log t)^{m_1-1}}$$

if $t < 0$, t can be replaced by $|t|$,

$$\therefore L(|t|^\lambda) = P_R(\lambda)|t|^\lambda, \quad t \neq 0$$

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theorem: $x_{ij}(t) = |t|^{\lambda_j} [\log|t|]^{i-1}$, $i = 1, \cdots, m_j$, $j = 1, \cdots, k$

$x_{ij}(t)$ are linearly independent, $t \neq 0$

proof:

let $s = \log|t|$, $\therefore |t| = e^s$

$$x_{ij}(t) = e^{s\lambda_j} s^{i-1}$$

if they are not independent,

$$\sum_{i,j} c_{ij} x_{ij} = 0$$

$$e^{s\lambda_1}(c_{11} + c_{21} s + \cdots + c_{m_1 1} s^{m_1 - 1})$$

$$+ \cdots + e^{s\lambda_k}(c_{1k} + c_{2k} s + \cdots + c_{m_k k} s^{m_k - 1}) = 0$$

$$\therefore e^{s\lambda_1} P_1(s) + e^{s\lambda_2} P_2(s) + \cdots + e^{s\lambda_k} P_k(s) = 0$$

consider the operator $D - \lambda_k$

$$(D - \lambda_k) e^{s\lambda_i} P_i(s) = (\lambda_i - \lambda_k) e^{s\lambda_i} P_i(s) + e^{s\lambda_i} P_i'(s)$$

$$\deg[P_i'(s)] < \deg[P_i(s)] = m_i - 1$$

if applying this operator to $e^{s\lambda_k} P_k(s)$ m_k times,

$$(D - \lambda_k)^{m_k} e^{s\lambda_k} P_k(s) = 0$$

thus, by applying $\prod_{i \neq I} (D-\lambda_i)^{m_i}$,

$$\prod_{i \neq I} (\lambda_I - \lambda_i)^{m_i} e^{\lambda_I s} P_I(s) + e^{\lambda_I s} Q(s) = 0$$

since $\deg[P_I(s)] > \deg[Q(s)]$,

if, in prior, $\deg[P_I(s)]$ is d_I, and it is the largest degree that is non-zero, this leads to contradiction. $\square$

The second-order equation

$$t^2 x'' + a(t) t x' + b(t) x = 0$$

$a(t) = \sum a_k t^k$

$b(t) = \sum b_k t^k$, $\quad |t| < \rho$

indicial polynomial $P_R(\lambda) = \lambda(\lambda-1) + a_0 \lambda + b_0$

let λ_1, λ_2 be eigenvalues, and $Re(\lambda_1) > Re(\lambda_2)$